The Association

of

Engineering and Shipbuilding Draughtsmen.

THE CRITICAL OR WHIRLING SPEED OF SHAFTS.

By WILLIAM A. NOBLE.

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THE CRITICAL OR WHIRLING SPEED OF SHAFTS.

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INTRODUCTION

THE problem of the whirling of shafts has received the attention of many eminent engineers. Consequently the information available is voluminous and the curious find themselves confronted with the task of perusing many works. Many of these writings are not easily accessible and also contain much information which is of purely academic interest. In view of these considerations the author has been prompted to write this pamphlet and it is hoped that sufficient information has been provided to enable the practical draughtsman to approach any critical speed problem with confidence.

In the preparation of a pamphlet the tendency is to view the subject from the point of view of the industry in which one is engaged. In this case, although the author's experience is with the marine steam turbine, it is emphasised that the theory is quite general and can be applied to any shaft. Thus those in the aircraft and electrical industries will find that the text is applicable to the problems encountered in these branches of engineering.

It is a primary consideration in the design of any power unit, be it rotary or reciprocating, that the periodic disturbing forces set up when the machine is in operation do not have a frequency in the neighbourhood of the natural frequency of the moving parts, or alternatively, if the frequencies do approach each other that they be out of phase and hence damp out.

Let it be supposed that a body is vibrating naturally in the manner shown in Fig. 1 (a)—ordinates represent amplitude and abscissae time—and in addition to the vibration (a) it is also subjected to the forced vibration shown at (b). The resultant vibration obtained by adding the ordinates of (a) and (b) algebraically is shown at (c). Since (a) is the natural frequency of vibration of the body and (b) is exactly in phase with it, we have the condition for resonance and consequently the amplitude of the combined vibration (c) will in theory become infinite; in practice, the value would be finite, due principally to the effect of elastic

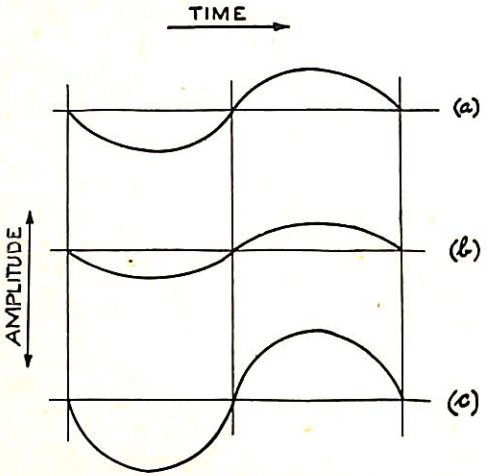


Fig. 1.

hysteresis in the material of the vibrating body, but the amplitude of the vibration may be such as to induce stresses which would cause failure.

Referring to Fig. 2, let (d) represent the natural frequency of vibration of a body and (e) a forced vibration exactly out of phase with that of (d). The resultant vibration is shown at (f) and the damping effect of the vibration (e) is clearly perceivable.

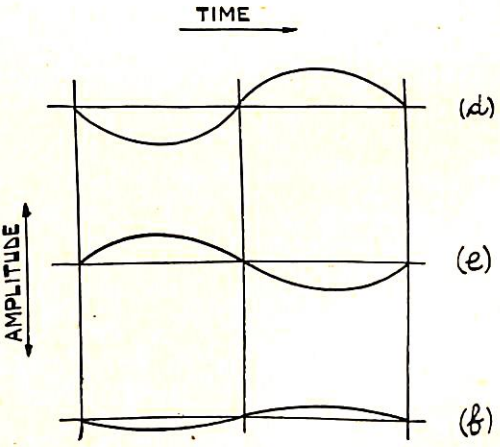


Fig. 2.

With regard to turbine rotors it is the in phase vibrations (Fig. 1) which must be guarded against, and it will be appreciated that it is imperative that the rotor must be run at such a speed that the forced vibration set up by the eccentricity of the mass centres does not approach the critical or natural frequency of vibration of the shaft. In reaction turbines, the small clearances of both axial and radial clearance blading indeed precludes the existence of any appreciable vibration and in view of these considerations it is the practice to run the rotors of large turbines somewhere in the region of 30% to 50% below the critical value.

The critical speed of a turbine rotor may be defined from consideration of the following—

In any shaft, be it in a horizontal or vertical position, it is impossible to have the centres of mass of the parts situated in the axis of rotation. In the case of the horizontal shaft, the bending due to the weight of the shaft prevents the attainment of this condition and in the case of the vertical shaft there is never exact coincidence between the mass centres and the axis of rotation, due to inaccuracies which occur in the course of manufacture and which are incapable of being eliminated completely. As the shaft speeds up from rest, these eccentricities cause a centrifugal force to act on the shaft causing it to bend and ultimately a speed is attained at which the centrifugal force is equal to the elastic righting force exerted by the shaft. If this speed is maintained, the shaft will fail and it is said to have failed at the fundamental critical speed.

The unstable condition just described requires time to develop and if the shaft is rapidly accelerated through the fundamental critical speed period it will steady up again and can be run with safety. In small turbines, that of de Laval for example, this is the practice and in Section 9, higher critical speed values will be discussed briefly.

SIMPLE THEORY

The simplest case is that of a light shaft with a comparatively heavy wheel, as depicted in Fig. 3.

Assuming the shaft to be weightless,

Let W = weight of wheel—lbs.
ω = angular velocity—radians/sec.
y = deflection at wheel—inches.
P = lateral stiffness of shaft—lbs./inch deflection.
g = gravitational acceleration = 386·4 inches/sec.^{2}.

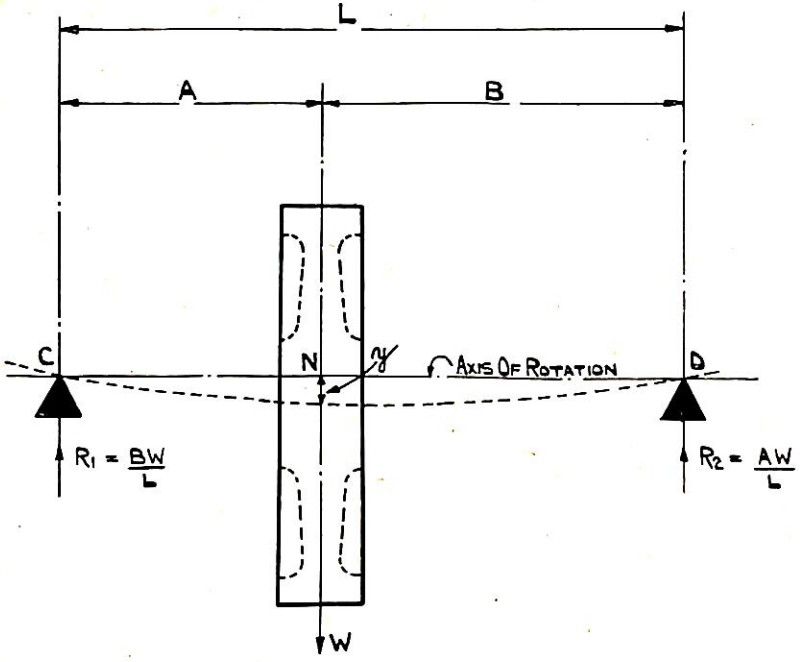


Fig. 3.

The eccentricity is y and the centrifugal force acting on the shaft is given by

$$F_c = \frac{W}{g} \cdot \omega^2 \cdot y \qquad (1)$$

The elastic righting force exerted by the shaft is given by

$$F_R = P \cdot y \qquad (2)$$

The condition for the critical speed is that $F_c = F_R$.

$$\therefore \quad \frac{W}{g} \cdot \omega^2 \cdot y = P \cdot y$$

Let ω_c = critical speed—radians/sec.
and n_c = critical speed—revs./min.

Then $$\omega_c = \sqrt{\frac{P \cdot g}{W}} \qquad (3)$$

and $$n_c = \sqrt{\frac{P \cdot g}{W}} \times \frac{30}{\pi} \qquad (4)$$

In order to evaluate n_c, it is necessary to determine the value of P.

Referring to Fig. 3, we proceed to evaluate P as follows :

The bending moment on the portion of the shaft to the left of W is, for any distance x from C as origin,

$$M_L = - \frac{B \cdot W \cdot x}{L}$$

and for the portion to the right of W is,

$$M_R = - \frac{B \cdot W \cdot x}{L} + W (x - A)$$

The equation for the deflection is,

$$y = \iint \frac{M}{E \cdot I} dx \cdot dx$$

where $I = \frac{\pi}{64} D^4$ for solid shafts and

$\frac{\pi}{64} (D^4 - d^4)$ for hollow shafts,

D and d being the shaft outside and inside diameters respectively.

For the left-hand portion of the shaft, the deflection equation expands to

$$y_L = - \frac{B \cdot W \cdot x^3}{6 L \cdot I \cdot E} + F \cdot x + G$$

and for the right-hand portion is

$$y_R = - \frac{B \cdot W \cdot x^3}{6 \cdot L \cdot I \cdot E} + \frac{W x^3}{6 I \cdot E} - \frac{W \cdot A x^2}{2 \cdot I \cdot E} + Hx + K$$

The four constants F, G, H and K are found from a knowledge of the boundary conditions.

Assuming the shaft to be simply supported, the boundary conditions are :

(a) The deflection is zero at C, *i.e.* when $x = O$.
(b) The deflection is zero at D, *i.e.* when $x = L$
(c) The deflections for the two portions are equal at N, *i.e.* when $x = A$.
(d) The slope of the elastic curve is the same at N for the two portions, *i.e.* when $x = A$.

The values of the constants are :

$$F = \frac{W \cdot A^3}{6L \cdot I \cdot E} - \frac{W \cdot L}{2 \cdot I \cdot E}\left(-\frac{B}{3} + \frac{L}{3} - A\right) - \frac{W A^2}{2 I \cdot E}$$

$$G = O$$

$$H = \frac{W \cdot A^3}{6L \cdot I \cdot E} - \frac{W \cdot L}{2I \cdot E}\left(-\frac{B}{3} + \frac{L}{3} - A\right)$$

$$K = -\frac{W \cdot A^3}{6 \cdot I \cdot E}$$

Hence the deflection for the left-hand portion is given by

$$y_L = \frac{W}{6 I \cdot E}\left\{\frac{B \cdot x\ (L^2 - x^2 - B^2)}{L}\right\} \qquad (5)$$

and for the right-hand portion is :

$$y_R = \frac{W}{6 I \cdot E}\left\{\frac{B \cdot x\ (L^2 - x^2 - B^2)}{L} + (x - A)^3\right\} \qquad (6)$$

The deflection at N by either (5) or (6) is :

$$y_N = \frac{W \cdot A^2 \cdot B^2}{3 \cdot I \cdot E \cdot L} \qquad (7)$$

When $y_N = 1$, $W = P$

$$\therefore \quad P = \frac{3 \cdot I \cdot E \cdot L}{A^2 \cdot B^2} \qquad (8)$$

Hence from (4),

$$n_c = \frac{30}{\pi}\sqrt{\frac{3 \cdot I \cdot E \cdot L \cdot g}{W \cdot A^2 \cdot B^2}} \qquad (9)$$

It may be easily shown that the critical speed is numerically equal to the natural frequency of vibration of the loaded shaft.

Referring again to Fig. 3, let it be supposed that the shaft is placed in a vertical position and let N be displaced a distance y horizontally and then released. The shaft will then vibrate with simple harmonic motion, and the amplitude of the (assumed undamped) vibration will be y at N.

Let n = frequency—oscillations/sec.

The kinetic energy of the wheel in the mean position is

$$\text{K.E.} = \frac{W}{2g}(2\ \pi \cdot y \cdot n)^2$$

At the extreme position, the kinetic energy is converted into strain energy.

The strain energy is $\frac{1}{2} P \cdot y^2$

Equating the strain energy to the kinetic energy,

$$\tfrac{1}{2}\ P\ .\ y^2 = \frac{W}{2g}\ (2\ \pi\ .\ y\ .\ n)^2$$

$$\therefore \quad n\ =\ \frac{1}{2\ \pi}\ \sqrt{\frac{P\ .\ g}{W}}$$

Let n_c = frequency—oscillations/min.

Then $$n_c\ =\ \frac{30}{\pi}\ \sqrt{\frac{P\ .\ g}{W}}$$

Substituting the value of P from (8),

$$n_c\ =\ \frac{30}{\pi}\ \sqrt{\frac{3\ I\ .\ E\ .\ L\ .\ g}{W\ .\ A^2.\ B^2}} \qquad (10)$$

Comparison of (9) and (10) proves the identity.

Example.—Estimate the critical speed of a shaft $\frac{3}{4}$" dia. and 27" long when it carries a load of 30 lbs. at a distance of 9" from one end. Neglect the weight of the shaft and assume it is simply supported on end bearings.

The values of the various quantities are :

I = ·0155 inch^4	E = 30 × 10^6 lbs./sq. inch.
L = 27"	g = 386·4 inches/sec.^{2}
W = 30 lbs.	A = 18"
	B = 9"

By formula (9) the critical speed is,

$$n_c\ =\ \frac{30}{\pi}\ \sqrt{\frac{3 \times \cdot0155 \times 30 \times 10^6 \times 386{\cdot}4 \times 27}{30 \times 324 \times 81}}$$

$$n_c \cong 1{,}295 \text{ R.P.M.}$$

It should be noted that although lb./inch units have been used in the foregoing, the theory is quite general and equally applicable to the metric system of units. Lb./inch units are used throughout the work, but the reader will find no difficulty in inserting the appropriate metric units should the need arise.

DUNKERLEY'S EMPIRICAL FORMULA.*

In the case of a shaft and disc rotor with a uniform shaft, an empirical method due to Professor Dunkerley, may be employed to calculate the critical speed. This method has been confirmed by experiment and may be applied safely to a two-bearing rotor supported on end bearings. It should, however, never be applied to a multi-bearing rotor as the results are liable to be very inaccurate.

* For a theoretical explanation of Dunkerley's formula, see "An Analytical Method," page 18.

If the shaft is not uniform, a method to be described later should be employed (page 12).

Let N = critical speed of whole rotor—revs./min.
n_1 = critical speed of shaft alone—revs./min.
n_2, n_3, etc. = critical speed due to each wheel mounted supposedly on a weightless shaft having the same lateral stiffness (the same dimensions) as the actual shaft—revs./min.

Then
$$\frac{1}{N^2} = \frac{1}{n_1^2} + \frac{1}{n_2^2} + \frac{1}{n_3^2} + \ldots \qquad (11)$$

The critical speed due to each wheel is found by the method already described. The critical speed of the shaft alone is found as follows :

Let w = weight of shaft—lbs./inch length.
E = Young's modulus—30×10^6 lbs./sq. inch.
ω = angular velocity at critical speed—rads./sec.
y = deflection at any distance x from a fixed origin—ins.
l = length of shaft between supports—inches.
g = gravitational acceleration = 386·4 inches/sec.^{2}
I = moment of inertia of shaft cross-section—inch.^{4}
n_1 = critical speed—revs./min.

The centrifugal force tending to bend the shaft is,

$$F_B = \frac{w}{g} \cdot \omega^2 \cdot y \text{ per unit length.}$$

The bending force is also given by

$$F_B = E \cdot I \frac{d^4y}{dx^4} \text{ per unit length.}$$

Hence
$$\frac{d^4y}{dx^4} = \frac{w \cdot \omega^2 \cdot y}{g \cdot E \cdot I}$$

Let
$$m^4 = \frac{w \cdot \omega^2}{g \cdot E \cdot I}$$

Then
$$\frac{d^4y}{dx^4} = m^4 \cdot y \qquad (12)$$

and
$$\omega = m^2 \sqrt{\frac{g \cdot E \cdot I}{w}} \qquad (13)$$

In order to evaluate m, it is necessary to partially solve equation (12).

Substituting $y = A \cdot e^{ax}$, where e = hyperbolic logarithm base, the auxiliary equation is,

$$m^4 = a^4$$

The roots of this equation are, $a = \pm m$ and $a = \pm i m$, where $i = \sqrt{-1}$.

$$\therefore \quad y = Ae^{mx} + B \cdot e^{-mx} + C \cdot e^{mxi} + D \cdot e^{-mxi}$$

This may be written,

$$y = F \sinh mx + G \cosh mx + H \sin mx + J \cos mx \qquad (14)$$

The boundary conditions for a simply supported shaft are,

(a) The deflection is zero at the supports, *i.e.* when $x = 0$ and $x = l$.

(b) The bending moment is zero at the supports and hence

$$\frac{d^2y}{dx^2} = 0 \text{ when } x = 0 \text{ and } x = l.$$

When $x = 0$ in (14), $G + J = 0$

Differentiating (14) twice and substituting $x = 0$, $G - J = 0$

Hence, $G = J = 0$.

When $x = l$ in (14), $F \sinh ml + H \sin ml = 0$

Differentiating (14) twice and substituting $x = l$

$$F \sinh ml - H \sin ml = 0$$

Hence, $2H \sin ml = 0$

If H is not zero, the shaft is vibrating and $\sin ml = 0$.

$$\therefore \quad ml = \pi, 2\pi, 3\pi, \text{ etc.}$$

The value $ml = 0$ is not admissable as the shaft would then be at rest.

Hence, $$m = \frac{\pi}{l}, \quad \frac{2\pi}{l}, \quad \frac{3\pi}{l}, \quad \text{etc.}$$

Substituting the values of m in (13), it will be seen that there are an infinite number of critical values and they are proportional to the numbers 1, 2^2, 3^2, 4^2, 5^2, etc.

Selecting the lowest value, the fundamental critical speed is,

$$\omega = \frac{\pi^2}{l^2} \sqrt{\frac{g \cdot E \cdot I}{w}} \qquad (15)$$

and $$n_1 = \frac{30}{l^2} \sqrt{\frac{g \cdot E \cdot I}{w}} \qquad (16)$$

Example.—Estimate the critical speed of a shaft 2″ dia. and 48″ long when it carries a load of 40 lbs. at a distance of 15″ from one end. Assume that the shaft is simply supported on end bearings and take the weight of a 2″ dia. shaft to be 0·9 lbs. per inch length.

The values of the various quantities are,

I	= ·7854 inch^4.	E	= 30×10^6 lbs./sq. inch.
L	= 48″	g	= 386·4 inches/sec.^{2}
W	= 40 lbs.	A	= 15″
B	= 33″	w	= ·9 lbs./inch.

By formula (16) the square of the critical speed of the shaft alone is,

$$n_1^2 = \frac{900\ \pi^2 \times 386{\cdot}4 \times 30 \times 10^6 \times {\cdot}7854}{48^4 \times {\cdot}9}$$

$$= 16962138{\cdot}7$$

By formula (9) the square of the critical speed of the load alone supposed mounted on a weightless shaft is,

$$n_2^2 = \frac{900 \times 3 \times {\cdot}7854 \times 30 \times 10^6 \times 48 \times 386{\cdot}4}{\pi^2 \times 40 \times 225 \times 1089}$$

$$= 12197384{\cdot}9$$

By formula (11) the critical speed N of the whole shaft is found from

$$\frac{1}{N^2} = \frac{1}{16962138{\cdot}7} + \frac{1}{12197384{\cdot}9}$$

and $N \cong 2660$ R.P.M.

THE TWO-BEARING ROTOR.

It has been pointed out by Lord Rayleigh that within wide limits different assumptions as to the shape of the deflection curve for a rod making free transverse vibrations do not materially affect the calculated value of the natural frequency. The correct value is a minimum and all others are in excess of this.

Advantage is taken of this fact in calculating the critical speed of turbine rotors. If the curve of static deflection is assumed to be the shape of the actual deflection curve when the shaft is at speed and the total resilience of the shaft is equated to the total kinetic energy in the mean position a close approximation to the actual critical speed will result. In the case of the two-bearing rotor, simply supported at its ends, the approximation is within 1% of the actual value and for all practical purposes this is sufficiently accurate.

Let w = variable load at any point along the shaft—lbs.
y = deflection at any point along the shaft—inches.
ω_c = critical speed—radians/sec.
n_c = critical speed—revs./min.
g = gravitational acceleration = 386·4 inches/sec.^{2}

The resilience of the shaft at any point is,

$$r = \tfrac{1}{2}\, w \,.\, y$$

The total resilience is,

$$R = \tfrac{1}{2}\, \Sigma\, w \,.\, y \qquad (17)$$

The kinetic energy for the mean position at any point is,

$$k = \frac{w}{2g} \cdot \omega_c^2 \cdot y^2$$

The total kinetic energy is,

$$K = \frac{\omega_c^2}{2g} \Sigma w \cdot y^2 \qquad (18)$$

Equating (17) and (18),

$$\tfrac{1}{2} \Sigma w \cdot y = \frac{\omega_c^2}{2g} \Sigma w \cdot y^2$$

then, $$\omega_c = \sqrt{\frac{\Sigma w \cdot y \cdot g}{\Sigma w \cdot y^2}} \qquad (19)$$

and $$n_c = \frac{30}{\pi} \sqrt{\frac{\Sigma w \cdot y \cdot g}{\Sigma w \cdot y^2}} \qquad (20)$$

The problem of a complex shaft may be treated graphically. A simple example of the method to be employed to find the curve of static deflection will suffice to render the reasoning apparent.

In Fig. 4, *a b* represents the line of action of a force F. It is required to determine the moment of F about a point *c*.

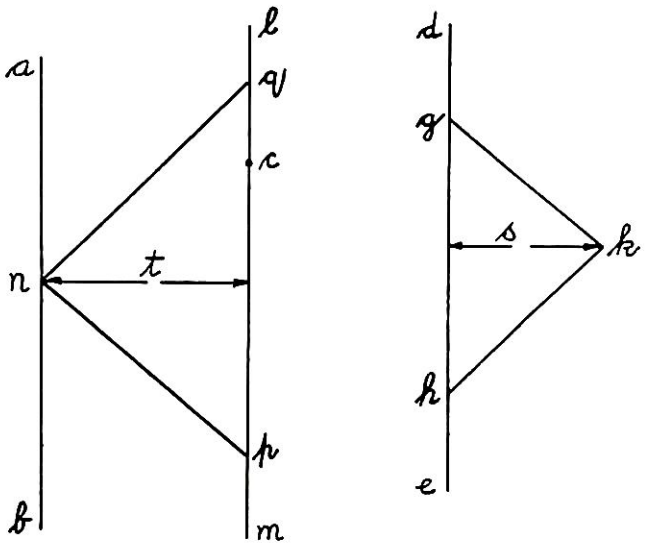


Fig. 4.

Draw $d\,e$ parallel to $a\,b$ and along $d\,e$, measure a distance $g\,h$ to represent the magnitude of F. Choose any pole k and join kg and kh. Through c draw lm parallel to ab. From any point n in ab draw np and nq parallel to kg and kh respectively. Then qp represents the moment of F about c if a suitable scale is employed for measuring its magnitude.

PROOF.—Since the triangles $n\,p\,q$ and $k\,g\,h$ are similar,

$$\frac{t}{pq} = \frac{s}{g\,h}$$

where t and s are the altitudes of the two triangles.

Hence, $$t\,.\,gh = s\,.\,pq \qquad (21)$$

Since gh represents the magnitude of F, then $t\,.\,gh$ is the moment of F about c, and if $s = 1$,

$$t\,.\,gh = pq$$

Thus with a suitable scale, pq is the moment of F about c.

SCALE.—Let $s = x$ times the linear unit employed. Let t be drawn to a scale where one linear unit represents y linear units and gh be drawn to a scale where one linear unit represents z units of force.

By (21) $$t\,.\,gh = s\,.\,pq$$

Applying the above scales,

$$t\,.\,y\,.\,z\,.\,gh = x\,.\,y\,.\,z\,.\,pq$$

Hence the scale of pq is $x\,.\,y\,.\,z$ units of moment per linear unit.

The foregoing construction is quite general and may be applied to the case of a complex shaft. The inclusion of such elementary reasoning will be justified if it eliminates the errors which often occur in arriving at the scale of the bending moment diagram.

The change of slope in radians between any two points on a beam of constant section is given by,

$$\frac{dy}{dx} = \frac{1}{E\,.\,I}\int_{x_1}^{x_2} M\,.\,dx$$

where M = bending moment.
E = Young's modulus.
I = moment of inertia of the section.
x = distance from a fixed origin.
y = deflection at distance x.

The quantity $\int_{x_1}^{x_2} M\,.\,dx$ represents the area of the bending moment diagram between the points considered.

Hence, $$\frac{\text{Actual area of B.M. diagram} \times \text{constant}}{E\,.\,I}$$

gives the change of slope in radians between the points considered.

The constant is found from a knowledge of the scales employed in drawing the bending moment diagram.

Let $1'' = y$ inches – length.
$1'' = z$ lbs. – force.

Let the polar distance $= x$ inches.

The scale of the ordinates of the bending moment diagram is as has been shown previously :

$$1'' = x \,.\, y \,.\, z \text{ lb. inch units.}$$

Hence one square inch of the bending moment diagram represents,

$$y\,(x \,.\, y \,.\, z) = x\, y^2\, z \text{ lb. inch}^2 \text{ units} = \text{constant.}$$

If the slope values are considered as imaginary loads acting through the centroids of the bending moment areas, the deflection diagram may be drawn in a similar manner to the bending moment diagram.

Referring to Fig. 4, let gh represent the change of slope between any two adjacent points on a beam. Then qp measured with a suitable scale represents the change of deflection between the points considered.

Let $1'' = y$ inches – length.
$1'' = z$ radians – slope.

Let the polar distance $= x$ inches.

Then the scale of the deflection diagram is

$$1'' = x \,.\, y \,.\, z \text{ inches.}$$

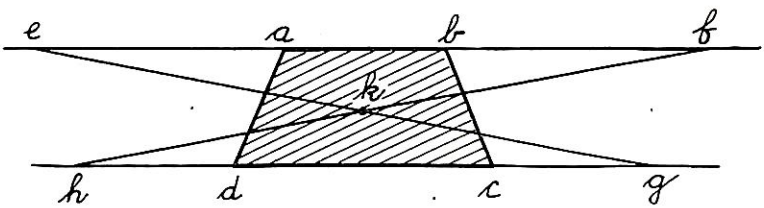


Fig. 5.

The bending moment areas with the exception of the triangular areas at the extreme ends approximate to a trapezoid. Fig. 5 illustrates a useful graphical method of determining the centroid of a trapezoid.

Let $a\ b\ c\ d$ represent any trapezoid. Produce ab and cd to e, f, g and h. Make ae and bf equal to cd and dh and cg equal to ab. Join eg and fh ; k is the centroid of $a\ b\ c\ d$.

In order to illustrate the method, an example has been worked in full and the procedure may be summarised in the following manner. Reference is made to Fig. 6 and Table 1.

Load Number.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Load – lbs.	28·5	43·5	171·5	171·5	427	427	427	427	427	427	427	427	805	187	117·5	117·5	43·5	28·5
B.M. Diagram Areas – Sq. Ins	·046	·107	·316	·478	·638	·828	·935	1·005	1·04	1·043	1·01	·943	·638	·71	·484	·316	·108	·046
I – Inch^4	62·8	200·3	1017·2	1017·2	5152·2	5152·2	5152·2	5152·2	5152·2	5152·2	5152·2	5152·2	1017·2	1017·2	489·1	489·1	200·3	62·8
Change Of Slope – Radians.	$\frac{·105}{10^4}$	$\frac{·077}{10^4}$	$\frac{·045}{10^4}$	$\frac{·068}{10^4}$	$\frac{·018}{10^4}$	$\frac{·023}{10^4}$	$\frac{·026}{10^4}$	$\frac{·028}{10^4}$	$\frac{·029}{10^4}$	$\frac{·03}{10^4}$	$\frac{·028}{10^4}$	$\frac{·026}{10^4}$	$\frac{·09}{10^4}$	$\frac{·1}{10^4}$	$\frac{·143}{10^4}$	$\frac{·093}{10^4}$	$\frac{·078}{10^4}$	$\frac{·105}{10^4}$
Deflection – Inches.	$\frac{·94}{10^4}$	$\frac{2·3}{10^4}$	$\frac{3·4}{10^4}$	$\frac{4·8}{10^4}$	$\frac{5·8}{10^4}$	$\frac{6·8}{10^4}$	$\frac{7·6}{10^4}$	$\frac{8·3}{10^4}$	$\frac{8·9}{10^4}$	$\frac{9·2}{10^4}$	$\frac{9·4}{10^4}$	$\frac{9·4}{10^4}$	$\frac{9·2}{10^4}$	$\frac{8·6}{10^4}$	$\frac{7·3}{10^4}$	$\frac{5·4}{10^4}$	$\frac{3·4}{10^4}$	$\frac{1·4}{10^4}$
$w.y$	·0027	·01	·058	·082	·248	·29	·325	·354	·38	·393	·401	·401	·741	·161	·086	·064	·015	·004
$w.y^2$	$\frac{·0025}{10^4}$	$\frac{·023}{10^4}$	$\frac{·2}{10^4}$	$\frac{·4}{10^4}$	$\frac{1·4}{10^4}$	$\frac{1·8}{10^4}$	$\frac{2·5}{10^4}$	$\frac{2·9}{10^4}$	$\frac{3·4}{10^4}$	$\frac{3·6}{10^4}$	$\frac{3·8}{10^4}$	$\frac{3·8}{10^4}$	$\frac{6·8}{10^4}$	$\frac{1·4}{10^4}$	$\frac{·63}{10^4}$	$\frac{·35}{10^4}$	$\frac{·051}{10^4}$	$\frac{·0056}{10^4}$

$\Sigma w.y = 4·0057$ $\qquad$ $\Sigma w.y^2 = ·0031.$

Table 1.

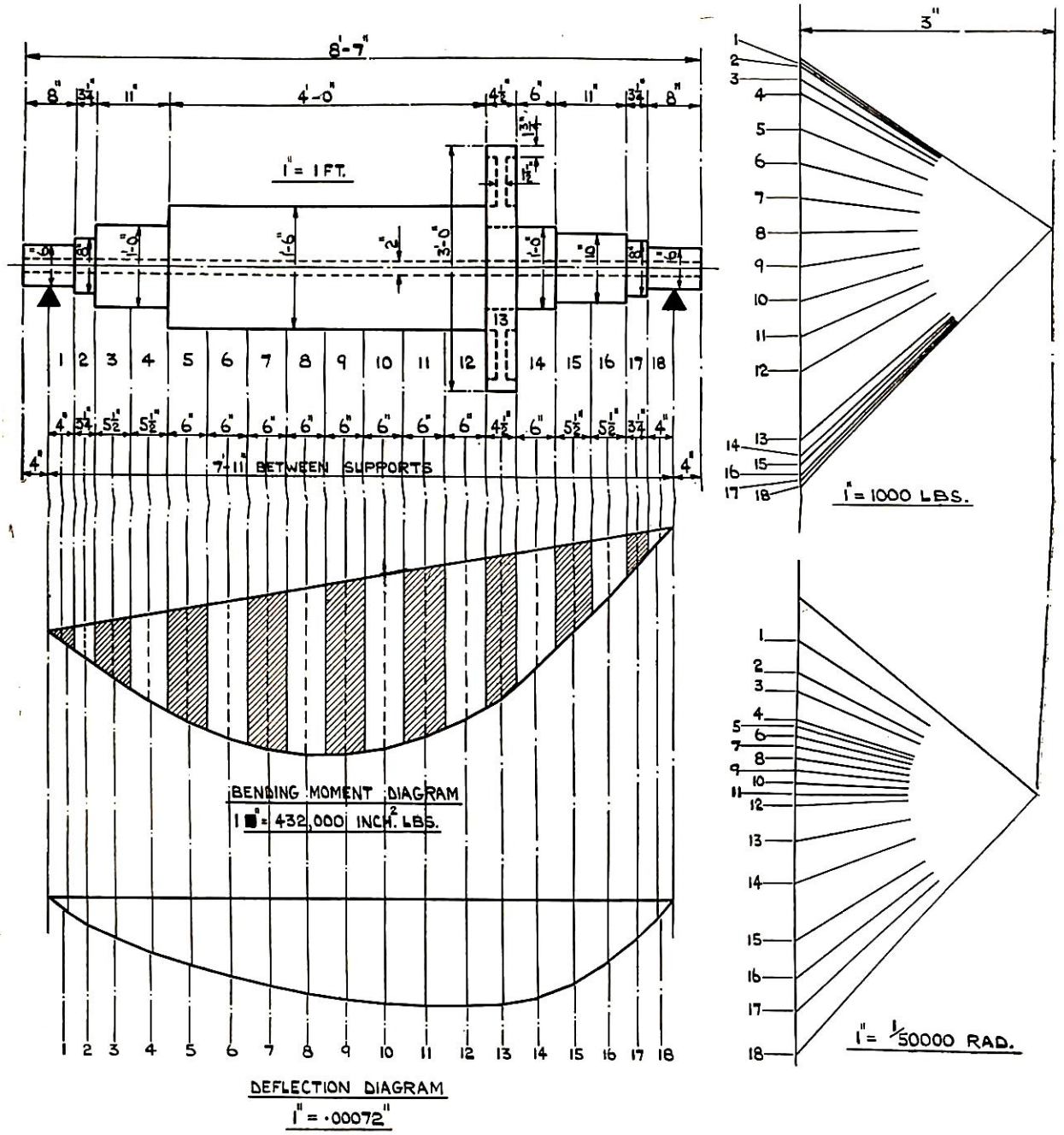

8'-7"
1" = 1 FT.
7'-11" BETWEEN SUPPORTS
BENDING MOMENT DIAGRAM
1 ■" = 432,000 INCH² LBS.
DEFLECTION DIAGRAM
1" = .00072"
3"
1" = 1000 LBS.
1" = 1/50000 RAD.

Fig. 6.

(a) Draw the shaft to as large a scale as practicable. (In the example the scale is 1″ = 1 ft., the scale being kept small to facilitate reproduction).

(b) Divide the shaft into a number of sections, I having a constant value over each section and estimate the weight of each; blading, etc., must be included. Let these weights act through the centres of mass of the sections.

(c) Draw the force polygon, using as large a force scale as practicable. In Fig. 6, the scale is 1″ = 1000 lbs. and the polar distance is 3″.

(d) Project the load lines vertically downwards, and by drawing lines parallel to the rays of the force polygon to intercept the load verticals, construct the bending moment diagram.

(e) Estimate the areas of the sections of the bending moment diagram under each load (sq. ins.).

(f) Calculate the change of slope over each section from the equation,

$$\alpha = \frac{\text{Actual area of B.M. diagram} \times \text{constant}}{E \cdot I}$$

The value of the constant is,

$$3 \times 12^2 \times 1000 = 432{,}000 \text{ inch}^2 \text{ lbs.}$$

(g) Using the slope values, draw the imaginary force polygon. The scale here is, 1″ = 1/50000 radian.

(h) Draw verticals through the centroids of the bending moment areas and from the imaginary force polygon construct the deflection diagram.

The scale of the deflection diagram is :

$$1'' = 3 \times 12 \times 1/50000 = \cdot 00072 \text{ inch.}$$

(j) Estimate the deflection at each load and calculate the values of $w \cdot y$ and $w \cdot y^2$. The various values are given in Table 1.

(k) Calculate the critical speed from formula (20). In the example, this is :

$$n_c = \frac{30}{\pi} \sqrt{\frac{4 \cdot 0057\, g}{\cdot 0031}}$$

$$\cong 6750 \text{ R.P.M.}$$

It should be noted that this is a first approximation, but, as has been mentioned already, is sufficiently accurate for a shaft of this type.

A more accurate value may be obtained by drawing a new deflection curve, using the original deflection curve as an assumed one, calculating the centrifugal force at each load about its respective radius of rotation (the respective deflection y on the assumed

deflection curve) and using these centrifugal forces instead of the original static loads. Formula (20) then takes the form,

$$n_k = \frac{30}{\pi} \sqrt{\frac{\Sigma P \cdot y^1 \cdot g}{\Sigma w \cdot (y^1)^2}} \qquad (20A)$$

where P = centrifugal loading at the speed n_c.
y^1 = deflection on the derived deflection curve due to the load P.

AN ANALYTICAL METHOD.*

The semi-graphical method of calculation just described will undoubtedly appeal to many, but some may prefer an analytical method. In any case an alternative method provides a ready means of checking the result obtained by the graphical method and also the method about to be described is not so laborious.

The problem of determining the critical frequency reduces to that of finding the deflection at various points of a beam when it is subjected to some particular system of loading. The system of loading as far as the critical speed calculation is concerned is dependent on the manner in which the shaft is supported.

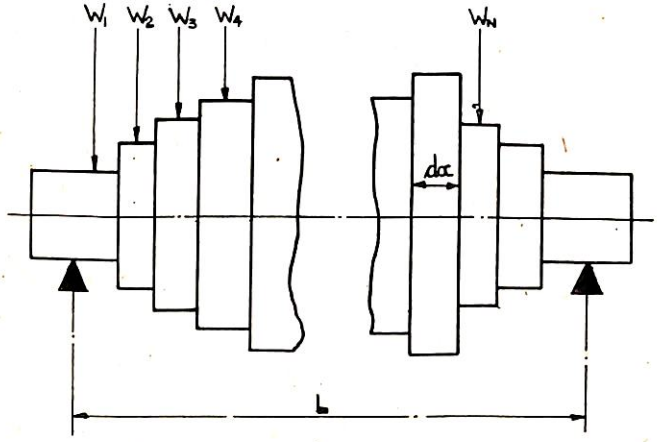


Fig. 7.

Consider Fig. 7.

Let y_N = deflection at load W_N.

Then the work performed by W_N when the beam is deflected a distance y_N at this point is $W_N/2 \cdot y_N$.

* "The Critical Speed of a Turbine Rotor," by J. Morris and S. J. E. Moyes. *Engineering*, 30th June, 1939.

Let W_N be increased to $(W_N + 1)$. The work done will then be $\frac{1}{2}(W_N + 1)(y_N + y)$, where y is the deflection due to the unit load acting alone in the position of W_N.

$$\frac{1}{2}(W_N + 1)(y_N + y) = \frac{W_N \cdot y_N}{2} + \frac{W_N \cdot y}{2} + \frac{y_N}{2} + \frac{y}{2}$$

but $y = y_N / W_N$

$\therefore \quad y_N = W_N \cdot y$

Hence the work done is

$$\frac{W_N \cdot y_N}{2} + y_N + \frac{y}{2} \tag{21}$$

This work is stored as energy by the shaft.

Let M = bending moment at any point of the shaft due to the normal loading W_N.

m = bending moment at any point of the shaft due to unit load acting alone at the position of W_N.

E = Young's modulus.

I = variable moment of inertia of the shaft cross-section.

The change of slope of the deflection curve for the element dx is,

$$\frac{(M + m)}{E \cdot I} dx$$

The energy stored due to this change of slope is,

$$\frac{(M + m)^2}{2\,E \cdot I} dx$$

The total energy stored due to the load $(W_N + 1)$ acting through the distance $(y_N + y)$ is,

$$\int_0^L \frac{(M + m)^2}{2\,E \cdot I} dx$$

$$= \int_0^L \frac{M^2}{2\,E \cdot I} dx + \int_0^L \frac{M \cdot m}{E \cdot I} dx + \int_0^L \frac{m^2}{2\,E \cdot I} dx$$

Consideration will show that,

$$\frac{W_N \cdot y_N}{2} = \int_0^L \frac{M^2}{2\,E \cdot I} dx$$

and $$\frac{y}{2} = \int_0^L \frac{m^2}{2\,E \cdot I} dx \tag{22}$$

hence $$y_N = \int_0^L \frac{M \cdot m}{E \cdot I} dx \tag{23}$$

If the sign of integration is replaced by the symbol Σ in (22) or (23) we have the approximations :

$$y = \sum_{o}^{N} \frac{m^2}{E \cdot I} dx \qquad (24)$$

$$y_N = \sum_{o}^{N} \frac{M \cdot m}{E \cdot I} dx \qquad (25)$$

where N is the number of positions considered.

Thus y_N can be calculated to any required degree of accuracy by splitting the shaft into short lengths and calculating the mean values of M, I and m for each length. The deflections thus found may be substituted in formula (20) and the critical speed calculated as before.

In order to simplify the work, a formula for the critical speed was derived from the rational explanation of Dunkerley's empirical formula.

The method of explanation of Dunkerley's formula used here is due to Hahn* and in order to render the reasoning apparent a simple example is taken ; the result, however, is quite general.

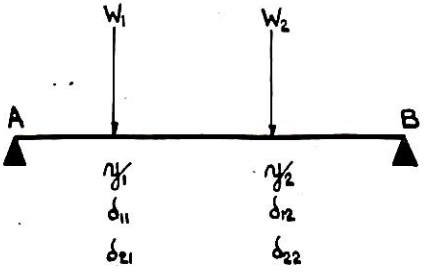


Fig. 8.

Let any shaft AB, Fig. 8, be loaded, as shown, and let the critical angular velocity of the shaft be ω_k.

Let the deflection at the critical speed in way of each load be y_1 and y_2. The centrifugal loading then is,

$$\frac{W_1}{g} \cdot \omega_k^2 \cdot y_1 \text{ and } \frac{W_2}{g} \cdot \omega_k^2 \cdot y_2$$

Let the shaft be at rest and let all the loading be removed. Then unit load in the position W_1 will produce the deflections δ_{11} and δ_{12} and unit load in the position W_2 will produce the deflections δ_{21} and δ_{22}.

* "Steam and Gas Turbines." Stodola. Page 1110.

Hence the deflections at the critical speed are given by,

$$y_1 = \frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{11} \cdot y_1 + \frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{21} \cdot y_2$$

$$y_2 = \frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{12} \cdot y_1 + \frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{22} \cdot y_2$$

These may be written

$$\left(\frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{11} - 1\right) y_1 + \frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{21} \cdot y_2 = 0$$

$$\frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{12} \cdot y_1 + \left(\frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{22} - 1\right) y_2 = 0$$

The eliminant of the equations for finite values of y_1 and y_2 is given by the determinant,

$$\begin{vmatrix} \left(\frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{11} - 1\right) & + \frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{21} \\ \frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{12} & + \left(\frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{22} - 1\right) \end{vmatrix} = 0.$$

Expanding we have,

$$\frac{W_1}{g} \cdot \frac{W_2}{g} \cdot \omega_k^4 \cdot \delta_{11} \cdot \delta_{22} - \frac{W_1}{g} \cdot \omega_k^2 \cdot \delta_{11} + 1 - \frac{W_2}{g} \cdot \omega_k^2 \cdot \delta_{22} - \frac{W_1}{g} \cdot \frac{W_2}{g} \cdot \omega_k^4 \cdot \delta_{21} \cdot \delta_{12} = 0$$

Approximating to the first order of small quantities,

$$\frac{1}{\omega_k^2} = \frac{W_1}{g} \cdot \delta_{11} + \frac{W_2}{g} \cdot \delta_{22} \qquad (26)$$

If in the example we consider the critical angular velocity for the loads acting singly, then for load W_1,

$$\left(\frac{W_1}{g} \cdot \omega_1^2 \cdot \delta_{11} - 1\right) y_a = 0$$

and for finite deflections,

$$\frac{W_1}{g} \cdot \omega_1^2 \cdot \delta_{11} - 1 = 0$$

hence,

$$\frac{1}{\omega_1^2} = \frac{W_1}{g} \cdot \delta_{11} \qquad (27)$$

Similarly,
$$\frac{1}{\omega_2^2} = \frac{W_2}{g} \cdot \delta_{22} \tag{28}$$

Substituting the values of (27) and (28) in (26) we have,
$$\frac{1}{\omega_k^2} = \frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} \tag{29}$$

which is the equation of Dunkerley.

Equation (26) may be written,
$$\frac{1}{\omega_k^2} = \Sigma \frac{W}{g} \cdot \delta$$
$$\therefore \quad \omega_k = \sqrt{\frac{g}{\Sigma W \cdot \delta}}$$

Let n_c = critical speed – revs./min.
$$n_c = \frac{30}{\pi} \sqrt{\frac{g}{\Sigma W \cdot \delta}} \tag{30}$$

The values of δ can be calculated from
$$\delta = y = \Sigma_0^N \frac{m^2}{E \cdot I} dx \tag{24}$$

Equation (24) can be written in a form which lends itself to systematic tabulation. The bending moment for unit load at Z, Fig. 9 for a beam of length L is $\frac{a_z \cdot b_z}{L}$ at Z.

The bending moment for all points to the left of Z is given by $a \cdot b_z/L$ and for all points to the right by $a_z \cdot b/L$. Inserting these values in (24) and taking due note of the limits we have,
$$\delta_z = \frac{b_z^2}{E \cdot L^2} \Sigma_1^z \frac{a^2 \cdot dx}{I} + \frac{a_z^2}{E \cdot L^2} \Sigma_{z+1}^N \frac{b^2 \cdot dx}{I}$$

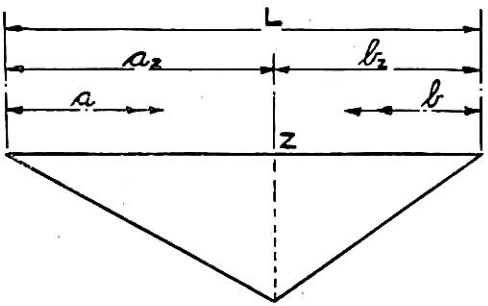


Fig. 9.

Further,

$$W_z \cdot \delta_z = \frac{W_z \cdot b_z^2}{E \cdot L^2} \sum_1^z \frac{a^2 \cdot dx}{I} + \frac{W_z \cdot a_z^2}{E \cdot L^2} \sum_{z+1}^s \frac{b^2 \cdot dx}{I} \quad (31)$$

Also formula (25) may be written,

$$y_z = \frac{b_z}{E.L} \sum_1^z \frac{M \cdot a \cdot dx}{I} + \frac{a_z}{E.L} \sum_{z+1}^s \frac{M \cdot b \cdot dx}{I} \quad (32)$$

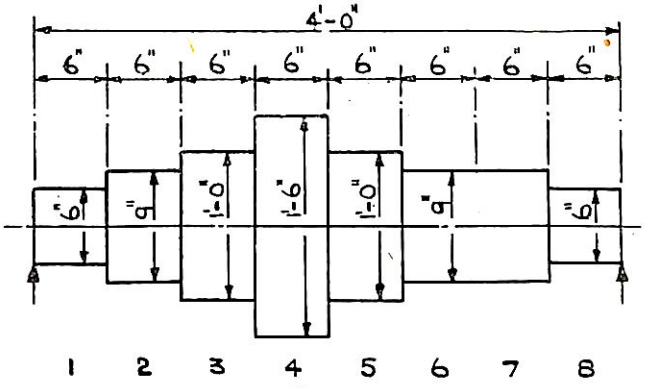


Fig. 10.

The critical speed of the shaft of Fig. 10 has been calculated by formula (30) using the values of (31) and also by formula (20) using the values of (32). The values of the various quantities for the two methods are given in Tables 2 and 3 respectively.

The critical speed by (30) is 17,170 R.P.M. and by (20) is 17,680 R.P.M. Assuming the latter value to be exact, that given by (30) is about 3% too low. The value given by (30) is always on the low side and may be as much as 5% removed from the actual value.

THE OVERHUNG SHAFT.

The overhung shaft is seldom encountered in marine turbines, but it is met with in aircraft gas turbines and certain types of electrical generating plant, such as water wheel driven alternators.

The procedure is essentially the same as that for the two-bearing rotor, but there are a few points which require special mention.

Referring to Fig. 11, it will be seen that the deflection curve dips between the supports and rises outside the right-hand support at a particular instant when the shaft is at speed. It is important to note that this is the condition which gives the lowest or fundamental critical speed, that is, a two-node deflection curve for the shaft, and it will be appreciated that in order to attain this the

Load Number	1	2	3	4	5	6	7	8
W — Lbs.	48	108	192	432·5	192	108	108	48
a — Inches	3	9	15	21	27	33	39	45
b — Inches	45	39	33	27	21	15	9	3
dx — Inches,	6	6	6	6	6	6	6	6
I — Inch^4	63·62	322·1	1018	5153	1018	322·1	322·1	63·62
M — Inch Lbs.	1947	5553	8511	10317	9529·5	8588·5	4999·5	1762·5
$\frac{M.a.dx}{I}$	550·86	930·96	752·45	252·27	1516·48	527·96	3632·05	7479·96
$\frac{M.b.dx}{I}$	8264·54	4034·18	1655·38	324·35	1179·49	2399·77	838·17	498·67
$\sum \frac{M.a.dx}{I}$	550·86	1481·82	2234·27	2486·54	4003·02	9282·63	12914·68	20394·64
$\sum \frac{M.b.dx}{I}$	10999·99	6885·83	5240·45	4916·1	3736·61	1336·84	498·67	—
$\frac{b_2}{EL}\sum \frac{M.a\,dx}{I}$	$\frac{·172}{10^4}$	$\frac{·401}{10^4}$	$\frac{·512}{10^4}$	$\frac{·466}{10^4}$	$\frac{·584}{10^4}$	$\frac{·967}{10^4}$	$\frac{·807}{10^4}$	$\frac{·425}{10^4}$
$\frac{a_2}{EL}\sum \frac{M.b.dx}{I}$	$\frac{·228}{10^4}$	$\frac{·431}{10^4}$	$\frac{·546}{10^4}$	$\frac{·717}{10^4}$	$\frac{·701}{10^4}$	$\frac{·306}{10^4}$	$\frac{·135}{10^4}$	—
y — Inches.	$\frac{·4}{10^4}$	$\frac{·832}{10^4}$	$\frac{1·058}{10^4}$	$\frac{1·183}{10^4}$	$\frac{1·285}{10^4}$	$\frac{1·273}{10^4}$	$\frac{·942}{10^4}$	$\frac{·425}{10^4}$
Wy	·00192	·00899	·02031	·05116	·02467	·01375	·01017	·00204
Wy^2	$\frac{·768}{10^7}$	$\frac{7·476}{10^7}$	$\frac{21·492}{10^7}$	$\frac{60·528}{10^7}$	$\frac{31·704}{10^7}$	$\frac{17·502}{10^7}$	$\frac{9·583}{10^7}$	$\frac{·867}{10^7}$

$\Sigma Wy = ·13301$ $\Sigma Wy^2 = ·0000149917$

Table 2.

Load Number	1	2	3	4	5	6	7	8
W — Lbs	48	108	192	432·5	192	108	108	48.
a — Inches	3	9	15	21	27	33	39	45
b — Inches	45	39	33	27	21	15	9	3.
dx — Inches.	6	6	6	6	6	6	6	6
I — Inch^4.	63·62	322·1	1018	5153	1018	322·1	322·1	63·62
$\frac{a^2\,dx}{I}$	·849	1·509	1·326	·513	4·297	20·286	28·333	190·978
$\frac{b^2\,dx}{I}$	190·978	28·333	6·418	·849	2·599	4·191	1·509	·849.
$\sum \frac{a^2\,dx}{I}$	·849	2·358	3·684	4·197	8·494	28·78	57·113	248·091
$\sum \frac{b^2\,dx}{I}$	44·748	16·415	9·997	9·148	6·549	2·358	·849	—
$\frac{W_z \cdot b_z^2}{E.L^2} \sum \frac{a^2\,dx}{I}$	$\frac{·00119}{10^3}$	$\frac{·0050}{10^3}$	$\frac{·0112}{10^3}$	$\frac{·01914}{10^3}$	$\frac{·01041}{10^3}$	$\frac{·01012}{10^3}$	$\frac{·00723}{10^3}$	$\frac{·00155}{10^3}$
$\frac{W_z \cdot a_z^2}{E.L^2} \sum \frac{b^2\,dx}{I}$	$\frac{·00028}{10^3}$	$\frac{·00208}{10^3}$	$\frac{·00625}{10^3}$	$\frac{·02524}{10^3}$	$\frac{·01326}{10^3}$	$\frac{·00401}{10^3}$	$\frac{·00202}{10^3}$	—
$W_z \cdot y_z$ — Inches.	$\frac{·00147}{10^3}$	$\frac{·00768}{10^3}$	$\frac{·01737}{10^3}$	$\frac{·04438}{10^3}$	$\frac{·02367}{10^3}$	$\frac{·01413}{10^3}$	$\frac{·00925}{10^3}$	$\frac{·00155}{10^3}$

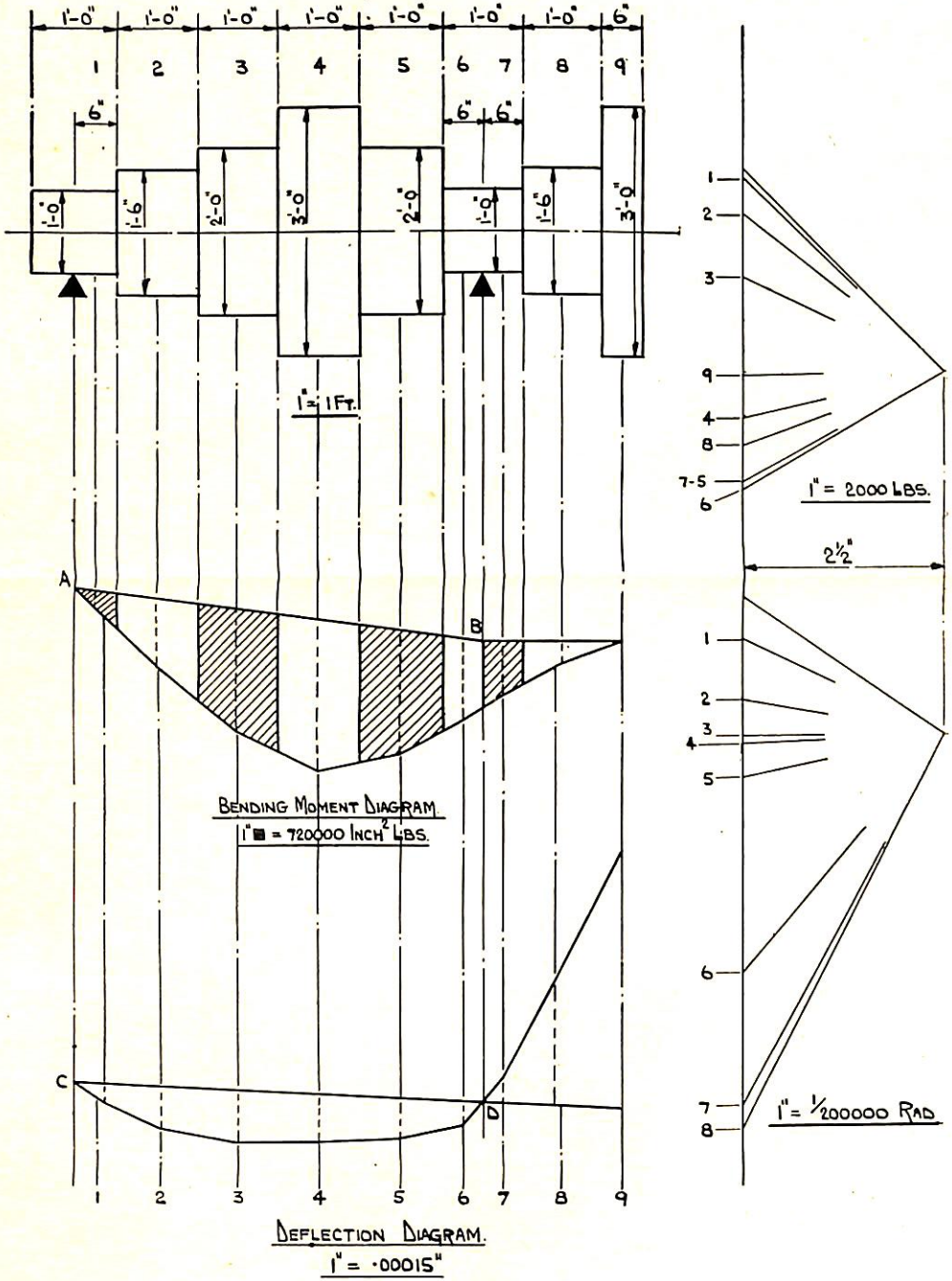
$$\sum Wy = ·0001195$$

Table 3.

Load Number.	1	2	3	4	5	6	7	8	9.
Load – Lbs.	192	865	1538	3460	1538	192	192	865	1730.
I – Inch^4	1018	5153	16,286	82,448	16286	1018	1018	5153.	
B.M. Areas — Sq. Ins.	·108	·8	1·42	1·71	1·39	·49	·33	·3	
Change Of Slope — Rads.	$\frac{\cdot 26}{10^5}$	$\frac{\cdot 37}{10^5}$	$\frac{\cdot 21}{10^5}$	$\frac{\cdot 05}{10^5}$	$\frac{\cdot 2}{10^5}$	$\frac{1\cdot 16}{10^5}$	$\frac{\cdot 78}{10^5}$	$\frac{\cdot 14}{10^5}$	
y – Inches	$\frac{\cdot 25}{10^4}$	$\frac{\cdot 75}{10^4}$	$\frac{\cdot 94}{10^4}$	$\frac{\cdot 84}{10^4}$	$\frac{\cdot 72}{10^4}$	$\frac{\cdot 44}{10^4}$	$\frac{\cdot 44}{10^4}$	$\frac{2\cdot 5}{10^4}$	$\frac{4\cdot 6}{10^4}$
Wy	·0048	·0649	·1446	·2906	·1107	·0084	·0084	·2163	·7958
Wy^2	$\frac{\cdot 12}{10^6}$	$\frac{4\cdot 868}{10^6}$	$\frac{13\cdot 592}{10^6}$	$\frac{24\cdot 41}{10^6}$	$\frac{7\cdot 97}{10^6}$	$\frac{\cdot 37}{10^6}$	$\frac{\cdot 37}{10^6}$	$\frac{54\cdot 075}{10^6}$	$\frac{366\cdot 068}{10^6}$

$\Sigma Wy = 1\cdot 6445.$ $\Sigma Wy^2 = \cdot 0004717$

Table 4.


1'-0"
1'-0"
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1'-0"
1'-0"
6"
1
2
3
4
5
6
7
8
9
6"
6"
6"
1'-0"
1'-6"
2'-0"
3'-0"
2'-0"
1'-0"
1'-6"
3'-0"
1" = 1 Ft.
A
B
C
D
Bending Moment Diagram
1" ■ = 720000 Inch² Lbs.
Deflection Diagram.
1" = ·00015"
1
2
3
9
4
8
7-5
6
1" = 2000 Lbs.
2½"
1
2
3
4
5
6
7
8
1" = 1/200000 Rad

Fig. 11.

loading inside and outside the supports must be assumed to act in opposite directions. Hence if the loads inside the supports be considered positive acting downwards, then those loads on the overhung portion must be considered negative acting upwards. It will thus be seen that it is necessary to have some notion of the shape of the deflection curve when the shaft is at speed and this is the case for all rotors except the simple two-bearing one of Fig. 6, where the loading is all of the same sense.

The force polygon is drawn as before, due note being taken of the signs of the loads. The bending moment diagram is constructed by taking the rays of the force polygon in order and drawing parallels to intercept the load lines. In the first instance the right-hand support line must be ignored and when the extreme right-hand load line is reached it will be found that the ray 9 is left over ; a line must be drawn parallel to this from the extreme right-hand load line to intercept the right-hand support line. The diagram is closed by joining AB.

The deflection diagram is drawn as before, the only difference being that the diagram is closed by the line CD drawn through the points of zero deflection.

The critical speed of the shaft of Fig. 11 has been worked out and the values of the various quantities are given in Table 4. The critical speed is by formula (20) 11,085 R.P.M. If any doubt should exist regarding the accuracy of the result the work will have to be repeated, using formula (20A). In general, when the overhung load is relatively heavy the latter calculation will have to be performed.

THE MULTI-BEARING ROTOR.

It is necessary before embarking on a discussion of the method to be employed in this case to state exactly what constitutes a multi-bearing rotor and what does not. A multi-bearing rotor may be defined as a continuous shaft or a series of shafts rigidly coupled together and supported on three or more bearings. If flexible couplings interpose the lengths between them must be considered as separate units for the purpose of the critical speed calculation.

The problem of the multi-bearing rotor reduces to that of finding the deflections at various points of a propped beam. The method of determining the deflections used here is that due to Wilson.*

Consider the beam ABCD, Fig. 12. Let the supports B and C be removed. The beam will deflect, as shown at (a) and the deflection at B is y_B and at C is y_C. Now, let all the loading be

* *Proc. Roy. Soc.*, vol. 62, Nov., 1897.

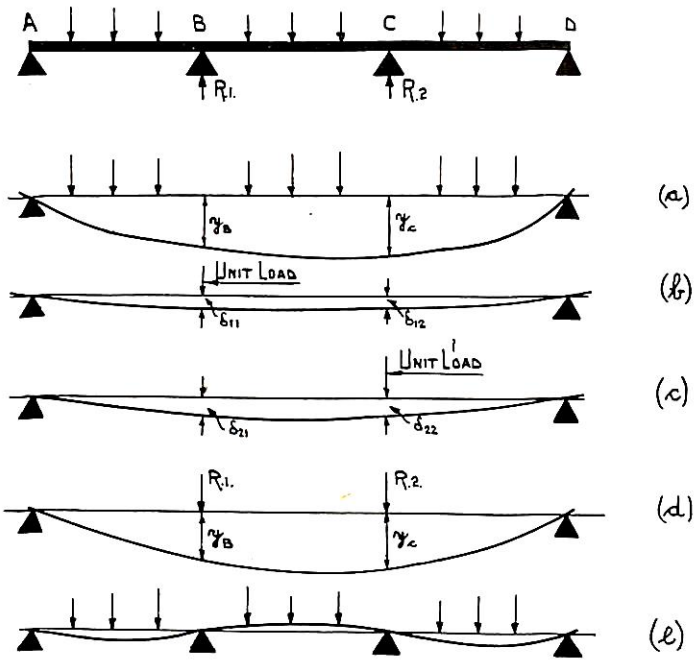

Fig. 12.

removed and unit load alone applied at B will produce the deflections δ_{11} and δ_{12} at B and C respectively (*b*). Repeat the process by applying the unit load at C and the deflections are δ_{21} at B and δ_{22} at C (*c*).

Let the reaction at the support B be R_1 and at C be R_2. Then, since the deflection of a beam is proportional to the load,

$$y_B = R_1 \cdot \delta_{11} + R_2 \cdot \delta_{21}$$
$$y_C = R_1 \cdot \delta_{12} + R_2 \cdot \delta_{22}$$

The above equations can be solved for R_1 and R_2, and if there are n intermediate supports there will be n such equations.

Let the beam be loaded at B and C with loads equal to R_1 and R_2 respectively. The deflection diagram is shown at *d*. If the deflections at *a* are considered positive and those at *d* negative and they are added algebraically, the actual deflection diagram of the beam will result *e*. It should be noted that the deflection diagram *d* can be constructed from the deflection diagrams *b* and *c* and the reactions R_1 and R_2.

In Fig. 12 the loads have all the same sense, whilst for the purpose of calculating the critical speed the loading is considered

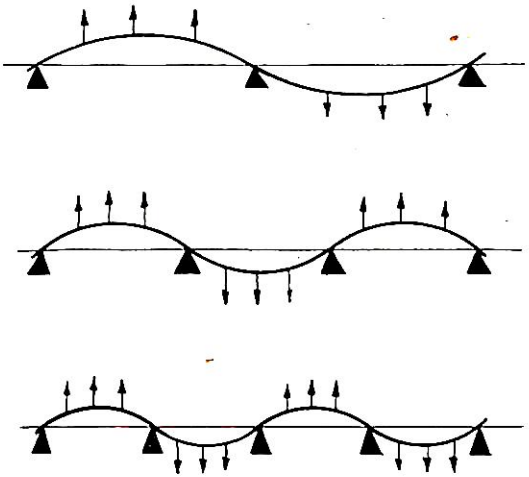

Fig. 13.

alternately positive and negative over the consecutive spans from consideration of the shape of the deflection curve at speed.

The shape of the deflection curve and the sense of the loading for a three, four and five bearing rotor is shown in Fig. 13.

It is unnecessary here to work an example, but the method may be summarised as follows :—

(a) Remove all the supports except the end ones and give the loads signs in conformity with Fig. 13.

(b) Draw the deflection diagram in the manner which has been described for the two-bearing rotor.

(c) Draw deflection diagrams for unit load acting alone at each of the removed support positions.

(d) From the general equation,

$$y = R_1 . \delta_{11} + R_2 . \delta_{21} + \ldots + R_N . \delta_{N1}$$

form N equations and solve for R_1, R_2 . . . R_N.

(e) Construct the deflection diagram for the loads R_1, R_2, . . . R_N acting on the shaft.

(f) Draw the actual deflection diagram for the supported shaft by subtracting that of e from that of b.

(g) Calculate the values of $W . y$ and $W . y^2$.

(h) By formula (20), calculate the critical speed of the shaft.

The value calculated by (20) must be considered in this case as a first approximation, and it is therefore necessary to repeat the work using formula (20A). If the derived critical speed is not

reasonably near to the original value (say 5% difference) the calculation will have to be performed a third time. It should be noted, however, that in practice it is seldom necessary to perform this third calculation.

THE GYROSCOPIC EFFECT OF THE ROTATING MASSES.

In the discussion up to the present it has been assumed that the masses have been concentrated at points situated in the axis of symmetry of the shaft ; it is apparent that this is not the case.

When the shaft is at speed a mass, such as a wheel, is inclined at an angle to the axis of rotation, the inclination depending on the slope of the deflection curve and consequently a moment is exerted by the wheel due to the tendency which it has to rotate in a plane perpendicular to the axis of rotation. This moment, according to its sign, tends to increase or decrease the deflection of the shaft and hence lower or raise the critical speed.

Cases may occur where the moment is appreciable ; a heavy wheel close to the bearing for example, or a wheel at the end of an overhung shaft, and it is therefore necessary to show how its effect may be taken into account.

In general, the gyroscopic moment has a decreasing effect on the deflection of the shaft and it is proposed herein to confine the discussion to this aspect of the problem.

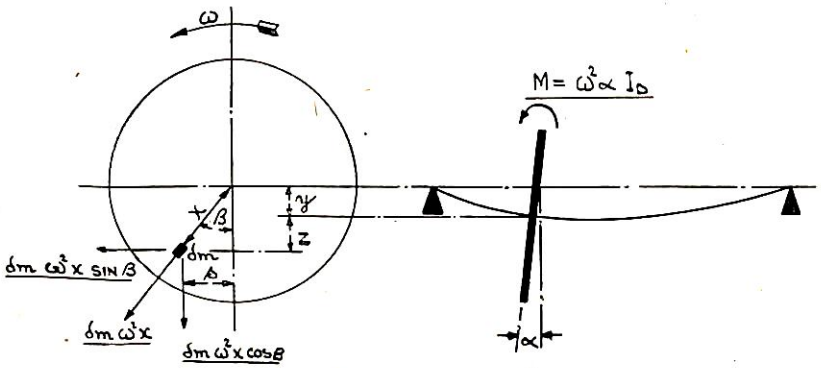

Fig. 14.

The wheel, shaft and deflection curve all rotate with the same sense and with the same angular velocity ω (Fig. 14), and consequently only the centrifugal forces are to be considered in calculating the gyroscopic moment.

The inclination of the wheel to the vertical is α radians and since this is a small quantity we may write,

$$\cos \alpha = 1 \text{ and } \sin \alpha = 0.$$

The centrifugal force for the element of mass δm is $\delta m \,.\, \omega^2 .\, x$. The vertical component of the centrifugal force is,

$$\delta F_v = \delta m \,.\, \omega^2 .\, x \cos \beta = \delta m \,.\, \omega^2 \,(y+z)$$

and the horizontal component is,

$$\delta F_H = \delta m \,.\, \omega^2 .\, x \sin \beta = \delta m \,.\, \omega^2 .\, s.$$

The total vertical force is,

$$F_v = \Sigma\, \delta m \,.\, \omega^2 \,(y+z) = \Sigma\, \delta m \,.\, \omega^2 .\, y + \Sigma\, \delta m \,.\, \omega^2 .\, z$$

but, $\Sigma\, \delta m \,.\, \omega^2 .\, z = 0$, since the simple moment of mass referred to a diameter passing through the centre of gravity of the wheel is zero.

The total vertical force is, therefore,

$$F_v = m \,.\, \omega^2 .\, y$$

The total horizontal force is,

$$F_H = \Sigma\, \delta m \,.\, \omega^2 .\, s = 0.$$

The moment of the centrifugal forces about a diameter situated in the centre of gravity plane of the wheel is,

$$\begin{aligned} M &= \Sigma\, \delta F_v \,.\, z \,.\, a \\ &= \Sigma\, \delta m \,.\, \omega^2 .\, y \,.\, z\, a + \Sigma\, \delta m \,.\, \omega^2 .\, z^2 .\, a \\ &= \Sigma\, \delta m \,.\, \omega^2 .\, z^2 .\, a \end{aligned}$$

The quantity $\Sigma\, \delta m \,.\, z^2$ is approximately the mass moment of inertia of the wheel referred to a diameter situated in the centre of gravity plane.

Hence, $$M = \omega^2 .\, a \,.\, I_D \qquad (33)$$

where, I_D = mass moment of inertia of the wheel referred to a diameter passing through the centre of gravity of the wheel.

The foregoing reasoning holds good for any of the rotating masses, be they wheels or portions of the drum.

It is a simple, if somewhat tedious, matter to take this moment into account. Let it be supposed that in the first instance the critical speed has been calculated, neglecting the gyroscopic moment. If the value obtained and the derived deflection diagram are taken as a basis for calculation, the gyroscopic moments can be computed.

Fig. 15 shows a two-bearing rotor having three wheels which exert moments M_1, M_2 and M_3. The length of the shaft is L. The force which must be exerted at the bearings in order to counteract the moments is given by

$$F = \frac{\Sigma\, M}{L}$$

where $\Sigma\, M$ is the algebraic sum of the moments M_1, M_2 and M_3. The ordinates of the bending moment diagram are found in the usual manner by taking the algebraic sum of the moments to the right or left of any given section. The sudden changes of the

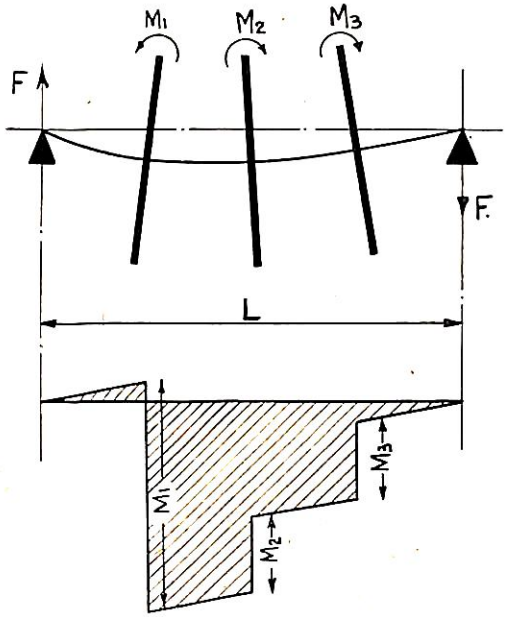

Fig. 15.

ordinates occur at the points of application of the moments M_1, M_2 and M_3, and it is therefore necessary when calculating the ordinates to first of all calculate the moment at each wheel, neglecting the moment exerted by the wheel and then to repeat the calculation taking the wheel moment into account.

The next step is to correct the bending moment diagram which has already been drawn in connection with the calculation for the critical speed when neglecting the gyroscopic moments by subtracting the diagram for the moments from it. The corrected bending moment diagram is then used to derive a new deflection diagram and the critical speed is calculated by formula (20). It will in general be unnecessary to repeat the calculation, using formula (20A) unless the gyroscopic effect is considerable.

The method of correction for a multi-bearing rotor is very similar to that just described and is briefly as follows :—

The critical speed is calculated neglecting the gyroscopic moments and then using the results as a basis of calculation, the gyroscopic moments are calculated.

Let it be supposed that the intermediate bearing supports are removed and a bending moment diagram for the gyroscopic moments (Fig. 16) constructed in the same manner as for the two-bearing rotor (Fig. 15).

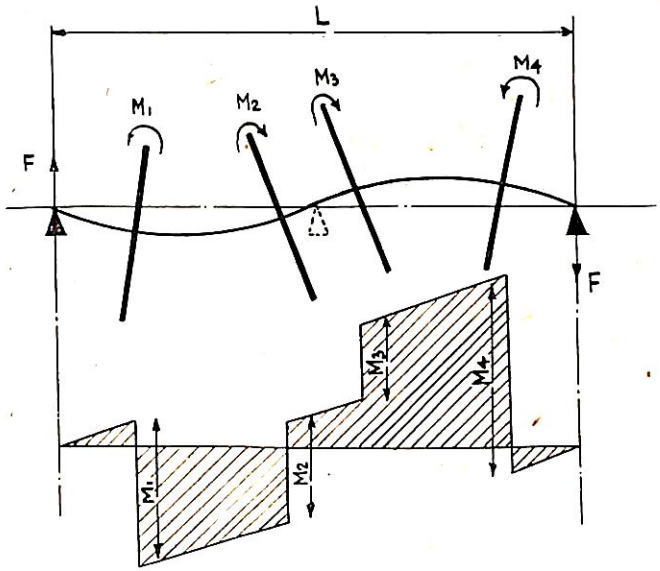

Fig. 16.

The bending moment diagram which has been drawn for the whole shaft when neglecting the gyroscopic moments and when the intermediate bearings are supposedly removed is corrected by subtracting that for the gyroscopic moments from it. The actual deflection diagram for the shaft is then derived by Wilson's method. The critical speed is calculated by formula (20A).

CRITICAL SPEEDS OTHER THAN THE FUNDAMENTAL.

Critical speeds other than the fundamental are seldom required except in connection with small high-speed turbines and certain categories of electrical generating plant, in which it is desirable and practicable to make the rotor as light as possible. In large installations, such as are used for ship propulsion, the rotors are run at speeds well below the fundamental critical value.

The actual shape of the elastic curve (the position of the nodes) for the higher critical values is a matter for conjecture in all but the simplest cases and hence formulae (20) and (20A), which give progressive approximations for the fundamental, are no longer convergent when applied to the higher categories.

It is necessary to have some conception of the shape of the shaft for the higher frequencies and Fig. 17 illustrates diagrammatically the probable shape of the elastic curve for a two-bearing rotor when running at the first, second and third critical speeds.

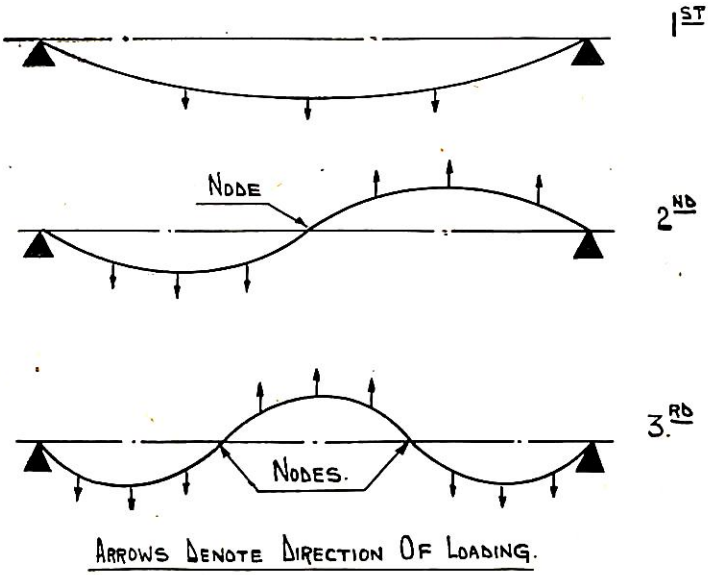

Fig. 17.

The method used here for computing the higher frequencies was suggested by Lord Rayleigh in his "Theory of Sound," and although the procedure will be found tedious, it gives very reliable results.

Let ω_1 = first critical angular velocity.
ω_2 = second critical angular velocity.
y_1 = deflection at any point of the shaft at the first critical speed.
y_2 = deflection at any point of the shaft at the second critical speed.
w = weight of shaft per unit length.
E = Young's modulus.
I = moment of inertia of the shaft cross-section (bending).
g = gravitational acceleration.
x = distances from a fixed origin.
L = length of shaft.

Considering the first critical speed, the condition for indifferent equilibrium is given by the differential equation,

$$\text{E.I.} \frac{d^4 y_1}{dx^4} = \frac{w}{g} \cdot \omega_1^2 \cdot y_1 \qquad (34)$$

Similarly, the condition for indifferent equilibrium at the second critical speed is,

$$\text{E.I.}\ \frac{d^4 y_2}{d x^4} = \frac{w}{g} \cdot \omega_2^2 \cdot y_2 \tag{35}$$

Multiply (34) by y_2 and subtract (35) multiplied by y_1; the whole is then multiplied by dx and the result summed over the length L of the shaft.

The result is,

$$\left.\begin{aligned} &\int_0^L \text{E.I.} \frac{d^4 y_1}{d x^4} \cdot y_2 \cdot dx - \int_0^L \text{E.I.} \frac{d^4 y_2}{d x^4} \cdot y_1 \cdot dx = \\ &\int_0^L \frac{w}{g} \cdot \omega_1^2 \cdot y_1 \cdot y_2 \, dx - \int_0^L \frac{w}{g} \cdot \omega_2^2 \cdot y_2 \cdot y_1 \, dx \end{aligned}\right\} \tag{36}$$

Considering the left-hand side of (36) and taking the integrals one at a time, we have, integrating by parts,

$$\left.\begin{aligned} &\int_0^L \text{E.I.} \frac{d^4 y_1}{dx^4} \cdot y_2 \cdot dx = \left[\text{E.I.} \frac{d^3 y_1}{dx^3} \cdot y_2\right]_0^L - \\ &\left[\frac{d_2 y_1}{dx^2} \cdot \frac{dy_2}{dx}\right]_0^L + \int_0^L \text{E.I.} \frac{d^2 y_1}{dx^2} \cdot \frac{d^2 y_2}{dx^2} \cdot dx \end{aligned}\right\} \tag{37}$$

Similarly the second integral expands to

$$\left.\begin{aligned} &- \int_0^L \text{E.I.} \frac{d^4 y_2}{dx^4} \cdot y_1 \cdot dx = - \left[\text{E.I.} \frac{d^3 y_2}{dx^3} \cdot y_1\right]_0^L + \\ &\left[\frac{d^2 y_2}{dx^2} \cdot \frac{dy_1}{dx}\right]_0^L - \int_0^L \text{E.I.} \frac{d^2 y_2}{dx^2} \cdot \frac{d^2 y_1}{dx^2} \cdot dx \end{aligned}\right\} \tag{38}$$

Inserting the values of (37) and (38) in (36), we have, since the quantities inside the brackets are zero from consideration of the boundary conditions for a freely supported shaft,

$$\int_0^L \frac{w}{g} \cdot \omega_1^2 \cdot y_1 \cdot y_2 \, dx - \int_0^L \frac{w}{g} \cdot \omega_2^2 \cdot y_2 \cdot y_1 \, dx = 0$$

$$\therefore \int_0^L w \cdot y_1 \cdot y_2 \, dx = 0 \tag{39}$$

Equation (39) gives the relationship between the deflections for the first and second critical speeds and any assumptions made must satisfy this relationship.

Equation (39) may be written

$$\sum_0^N \text{W} \cdot y_1 \cdot y_2 = 0 \tag{40}$$

where N = number of positions considered,
W = $w \cdot dx$ = weight of short lengths of the shaft.

The values of W and y_1 are known from the calculations for the first critical speed. Let an assumed deflection curve be drawn for the second critical speed in conformity with Fig. 17.

Let z = distance of the node from the fixed origin.
y = deflection at any point of the assumed curve.

The portion of the curve over the distance z is left unchanged and the ordinates of the remainder multiplied by a constant K chosen that the relationship (40) is satisfied. The constant K is found as follows :

From equation (40),

$$\Sigma_o^N W \cdot y_1 \cdot y_2 = \Sigma_o^z W \cdot y_1 \cdot y + K \Sigma_z^L W \cdot y_1 \cdot y = 0$$

$$\text{Hence } K = - \frac{\Sigma_o^z W \cdot y_1 \cdot y}{\Sigma_z^L W \cdot y_1 \cdot y} \qquad (41)$$

where the values of W, y_1 and y are read for the same points.

The corrected curve is then taken as a basis for calculation and a value for the second critical speed is computed by formula (20).

In order to determine the minimum value of the second critical speed, it is necessary to make at least three assumptions regarding the shape of the elastic curve. The three values of n_c thus calculated are plotted as a function of the distances z of the node points from the fixed origin. A smooth curve drawn through the three points will yield a minimum value of the second critical speed.

The procedure may be summarised as follows :—

(*a*) Draw the static deflection curve and calculate the fundamental critical speed by formula (20).

(*b*) Choose a point on the shaft and assume this is the position of the node. In this connection, experience alone is the guide to a suitable choice and it may occur that the point chosen will be unsuitable and the work will have to be repeated.

(*c*) Give the loads on each side of the node point signs in conformity with Fig. 17, and draw the deflection curve. In general, the position of the node on this derived deflection curve will not coincide with that first chosen, but if the two points are in the vicinity of each other, the derived deflection curve will be suitable for our purpose.

(*d*) Calculate the value of the constant K by formula (41), and hence correct the ordinates of the assumed deflection curve. It should be noted that if the corrected ordinates of the assumed curve are such as to produce a very decided discontinuity in the corrected curve the work will have to be repeated and a more acceptable position of the node chosen.

(*e*) Calculate a value of the second critical speed by formula (20).

(*f*) In the same manner, assume two more deflection curves and calculate the values of the second critical speed for each. It may be noted that the choice of the nodes here will be governed to a great extent by the results of the first assumption.

(*g*) Measure the distance of the nodes on the assumed deflection curves from some fixed origin and plot them against the corresponding critical values. The minimum value of the second critical speed is then read from the graph, which is drawn smoothly through these three points.

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